



# A characterization of the solutions of steady Hamilton-Jacobi equations

Gawtum Namah, Mohammed Sbihi

## ► To cite this version:

Gawtum Namah, Mohammed Sbihi. A characterization of the solutions of steady Hamilton-Jacobi equations. 2010. hal-00495318

**HAL Id: hal-00495318**

**<https://hal.science/hal-00495318>**

Preprint submitted on 25 Jun 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A characterization of the solutions of steady Hamilton-Jacobi equations\*

Gawtum Namah<sup>†</sup>, Mohammed Sbihi<sup>‡</sup>

## Abstract

In this note, we propose to revisit the approximate stationary Hamilton-Jacobi equations and analyse the corresponding solutions following certain properties of the hamiltonian. This enables us to give a characterization of the zero set of the limiting solution. We also remark that the analysis can be applied for evolution equations with a time periodic source term.

## 1 Introduction

In this note, we will be concerned with the viscosity solutions  $u_\alpha$  of the approximate Hamilton-Jacobi type equations

$$\alpha u + H(x, Du) = 0, \quad x \in \mathbb{R}^N \quad (1)$$

for  $\alpha > 0$  and with the characterization of these solutions and also of the limit of the sequence  $(u_\alpha)_\alpha$  as  $\alpha$  goes to zero. In the above, the hamiltonian  $H$  is a continuous function of its variables and  $Du$  denotes the gradient of  $u$ .

Now one knows that for  $\alpha > 0$ , equation (1) admits a unique viscosity solution under some suitable hypotheses on the hamiltonian  $H$ . We need notably a boundedness property *wrt* the space variable. For this sake, we will suppose that

$$H \text{ is periodic in } x. \quad (2)$$

We will also want some regularity of the solutions. This will be achieved by the coercivity condition

$$\lim_{|p| \rightarrow \infty} H(x, p) = \infty, \text{ uniformly in } x \in \mathbb{R}^N. \quad (3)$$

The above is a weak regularity assumption in the sense that no convexity assumption of  $H$  is called for. Under the hypotheses (2) and (3), for every  $\alpha > 0$ , equation (1) admits a unique viscosity solution  $u_\alpha \in W^{1,\infty}(\mathbb{R}^N)$ . Moreover  $u_\alpha$  is periodic. We refer the reader to [5], and [1] for a review of the definition and properties of viscosity solutions

---

\*Mathematics Subject Classifications: 35F25, 35B40, 35B10.

<sup>†</sup>Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray 25030 Besançon Cedex France. Email : namah@univ-fcomte.fr

<sup>‡</sup>ENAC, 7 avenue Edouard Belin 31055 Toulouse Cedex 4 France. Email : sbihi@recherche.enac.fr

for Hamilton-Jacobi equations.

The properties of  $u_\alpha$  will naturally depend on those imposed on the hamiltonian  $H$ . We want to inquire about this dependence. For this, we start by assuming that the function  $x \mapsto H(x, 0)$  has a sign. As a consequence, we have

LEMMA 1. Suppose  $H(x, 0) \leq 0$  for all  $x \in \mathbb{R}^N$ . Then (i)  $u_\alpha \geq 0$  for all  $\alpha > 0$  and (ii)  $(u_\alpha)_\alpha$  is increasing when  $\alpha$  goes to zero.

We thus need to bound  $(u_\alpha)$  if we want to say anything about its convergence. Notice also that, as we naturally expect the limit to be a solution of

$$H(x, Du) = 0, \quad x \in \mathbb{R}^N \quad (4)$$

we are brought to inquire about the well-posedness of (4) in the class of bounded continuous solutions. The latter solvability is not to be taken for granted. In fact, one can show that if the so-called ergodic equation

$$H(x, Du) = \lambda, \quad x \in \mathbb{R}^N \quad (5)$$

is solvable in the class of bounded solutions, then it is so for a unique ergodic constant  $\lambda \in \mathbb{R}$ . See for example the paper of Lions, Papanicolaou and Varadhan [6] where the above equation appears as the cell problem in a homogenization process. This unique  $\lambda$ , usually denoted by the letter  $c$ , is also called the critical value and the corresponding solutions, critical solutions, see for *e.g.* [3] and [4]. In fact the hypotheses (2) and (3) guarantee the solvability of (5) for some  $\lambda \in \mathbb{R}$  and thus we just demand that the critical value be equal to zero in our case. Before going further, let us give sufficient conditions for this to be true.

LEMMA 2. Let  $H$  satisfy (2) and (3) with  $H(x, 0) \leq 0$ ,  $x \in \mathbb{R}^N$ . Assume that there exists a point  $x^* \in \mathbb{R}^N$  such that  $H(x^*, p) \geq 0$  for all  $p \in \mathbb{R}^N$ . Then

(5) is solvable iff  $\lambda = 0$ .

A typical example will be the standard hamiltonian in classical mechanics which is given by

$$H(x, p) = |p|^2 - V(x)$$

with  $V$  periodic in  $x$  and  $\min V = 0$ . It is not difficult to see that  $H$  verifies the assumptions of the above lemma so that  $|Du|^2 - V(x) = \lambda$  is solvable only for  $\lambda = 0$ . We thus recover the result observed in [6] where the above example is treated rather in details. Let us also point out that the critical value  $c$  can be rather explicitly characterized under some convexity assumptions on  $H$ . One can see for example the paper by Fathi-Siconolfi [4], where quasiconvex hamiltonians are considered and the one by Roquejoffre in a 1-D context [8], where it is notably shown that in some cases, which includes a strict convexity assumption of  $H$  wrt  $p$ , we have

$$c = \max_x \min_p H(x, p).$$

We now state

LEMMA 3. Let  $H(x, 0) \leq 0$  and (4) be solvable. Then

$$u_\alpha \xrightarrow{\alpha \rightarrow 0} u_0 = \sup_{\alpha > 0} u_\alpha \quad \text{in } C^0(\mathbb{R}^N)$$

with  $u_0$  a periodic solution of (4).

Now we want to make more precise the characterization of the limit  $u_0$ . Define for this sake the set

$$Z = \{x \in \mathbb{R}^N : H(x, 0) = 0\}. \quad (6)$$

It is easy to verify that, due to the periodicity condition,  $Z$  is a nonempty set. We then propose

LEMMA 4. Under the hypotheses of Lemma 3,  $u_0$  attains its minimum on  $Z$  and vanishes at the minima points.

In fact an analogous property applies for  $u_\alpha$ . More precisely if  $x_0$  is a minimum point of  $u_0$  then it is a minimum point of  $u_\alpha$  for all  $\alpha > 0$  and  $u_\alpha(x_0) = 0$ . But of course, at this stage we cannot say that the minima points of  $u_0$  and those of the functions  $u_\alpha$  coincide, *i.e.* a point  $x_1$  can be a minimum point of a function  $u_\alpha$  but without being a minimum point of  $u_0$ .

Also Lemma 4 tells us that if  $x_0$  is a minimum point of  $u_0$  then  $x_0 \in Z$  and  $u_0(x_0) = 0$ . But does the zero set of  $u_0$  coincide with  $Z$ ? We now give our main result which states that under an additional condition on  $H$ , the zero set of  $u_0$  coincides with the zero set of  $u_\alpha$  for all  $\alpha > 0$  and is equal to the set  $Z$ .

THEOREM. Let  $H$  satisfy (2) and (3) with  $H(x, 0) \leq 0$ ,  $x \in \mathbb{R}^N$ . Assume also that  $H(x, p) \geq 0$  for all  $x \in Z$  and  $p \in \mathbb{R}^N$ . Then the sequence of solutions  $u_\alpha$  of (1) converges to a periodic solution  $u_0$  of (4) whose zero set is exactly equal to  $Z$ . Moreover  $Z$  coincides also with the zero set of  $u_\alpha$  for all  $\alpha > 0$ .

Note that the assumptions of the above theorem ensure the solvability of (4) via the Lemma 2. Let us point out that the above type of characterization is known for time asymptotic limits of Cauchy problems. See for *e.g.* [7] where the zero set of the limiting solution was shown to coincide with the maxima points of the initial condition. The analogy seems to stop here because in the case at present, there is of course no initial condition. What is important in both cases is some monotonicity character of the solutions, *wrt* the time variable for the Cauchy problem and *wrt*  $\alpha$  for the steady case as stated in Lemma 1.

We proceed in the next section by giving an application of the above result and notably show how the previous analysis can be profitably used to characterize the asymptotic limit in the case of evolution equations having a time periodic source term. And finally the last part will be devoted to the proofs of the lemmas and of the theorem.

## 2 The time periodic case

We consider therefore the following Hamilton-Jacobi type equation

$$\alpha u + \partial_t u + H(x, Du) = f(t), \quad (x, t) \in \mathbb{R}^N \times \mathbb{R} \quad (7)$$

where  $f$  is a continuous  $T$ -periodic function in  $t$ . One knows that under assumptions (2) and (3), equation (7) admits a unique  $T$ -periodic viscosity solution  $u_\alpha(x, t)$  in  $W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$ , see for example [2]. We enquire about the limit when  $\alpha$  goes to zero. To simplify the presentation and without loss of generality, we will suppose that the mean of  $f$  over one period is zero.

We proceed by a decoupling of the equation (7). Consider for this sake

$$\alpha\phi + H(x, D\phi) = 0, \quad x \in \mathbb{R}^N \quad (8)$$

and

$$\alpha v + v_t = f(t), \quad t \in \mathbb{R}. \quad (9)$$

The *ode* (9) admits also a unique periodic solution  $v_\alpha$  for every  $\alpha > 0$  and which has the explicit formulation

$$v_\alpha(t) = \frac{e^{-\alpha t}}{1 - e^{-\alpha T}} \int_0^T e^{-\alpha(T-s)} f(s) ds + \int_0^t e^{-\alpha(t-s)} f(s) ds.$$

It is then not difficult to verify that

$$v_\alpha(t) \xrightarrow{\alpha \rightarrow 0} F(t) + C_* \quad \text{for all } t \in \mathbb{R}$$

where

$$F(t) = \int_0^t f(s) ds \quad \text{and the constant } C_* = \frac{1}{T} \int_0^T s f(s) ds.$$

Now if we denote by  $\phi_\alpha(x)$  the solution of (8), then it is immediate to verify that  $u_\alpha(x, t) = \phi_\alpha(x) + v_\alpha(t)$  is the solution of (7). Therefore the study of the limit, when  $\alpha$  goes to zero, of the time periodic solution  $u_\alpha$  of (7) is reduced to that of the solution  $\phi_\alpha$  of the stationary equation (9). In fact, the limiting solution, which is time periodic via  $F$ , can thus be completely explicit as we have

$$u_\alpha(x, t) \xrightarrow{\alpha \rightarrow 0} \phi_0(x) + F(t) + C_* \quad \text{uniformly in } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

with  $\phi_0$  as previously characterized following the assumptions on the hamiltonian  $H$ .

### 3 Proof of the Theorem

We start by proving the preliminary lemmas.

#### 3.1 Proof of Lemma 1

Observe that as  $H(x, 0) \leq 0$ , 0 is a subsolution of (1). By comparison results, we then have  $u_\alpha \geq 0$ . Next take  $0 < \alpha \leq \beta$  and notice that  $u_\alpha$  is a supersolution of

$$\beta u(x) + H(x, Du) = 0, \quad x \in \mathbb{R}^N,$$

since we know that  $(\beta - \alpha)u_\alpha \geq 0$ . Again by comparison results, we have  $u_\alpha \geq u_\beta$ .

### 3.2 Proof of Lemma 3

We now suppose that (4) is solvable and let  $\phi$  be a continuous bounded solution. Then observe that  $\phi + \|\phi\|_\infty$  is a supersolution of (1) so that we have

$$0 \leq u_\alpha \leq \phi + \|\phi\|_\infty$$

*i.e.*  $(u_\alpha)_\alpha$  is bounded. Considering the monotonicity of the sequence  $(u_\alpha)_\alpha$ , we then have

$$u_\alpha(x) \xrightarrow{\alpha \rightarrow 0} u_0(x) = \sup_{\alpha > 0} u_\alpha(x), \quad x \in \mathbb{R}^N.$$

In fact, due to the coercivity condition (3),  $(u_\alpha)_\alpha$  is uniformly bounded in  $W^{1,\infty}(\mathbb{R}^N)$  so that the above convergence is uniform, at least locally. The stability result of viscosity solutions then yields that  $u_0$  is a viscosity solution of (4). The periodicity of  $u_0$  results from that of each  $u_\alpha$  and we are done with the proof of Lemma 3.

### 3.3 Proof of Lemma 4

To prove the lemma, first observe that the set  $Z = \{x \in \mathbb{R}^N : H(x, 0) = 0\}$  is non empty. Indeed at a minimum point  $x_0$  of  $u_0$  (its existence is guaranteed by the periodicity of  $u_0$ ) we have  $H(x_0, 0) \geq 0$  which together with the assumption  $H(x, 0) \leq 0$  leads to  $H(x_0, 0) = 0$ , *i.e.*  $x_0 \in Z$ . Note at this stage that  $H$  just bounded (instead of being periodic) would not have sufficed. It remains to show that  $u_0(x_0) = 0$ . We already know that  $u_\alpha \geq 0$  for all  $\alpha > 0$  and thus  $u_0(x_0) \geq 0$ . Now one can easily verify that  $u_0 - u_0(x_0)$  is a supersolution of (1). By comparison result, we then have  $u_0 - u_0(x_0) \geq u_\alpha$  for all  $\alpha > 0$ . A passage to the limit then leads to  $u_0(x_0) \leq 0$  and thus  $u_0(x_0) = 0$ . As pointed out before, since for all  $\alpha$  we have  $0 \leq u_\alpha \leq u_0$ ,  $x_0$  is also a minimum point for  $u_\alpha$  for all  $\alpha > 0$  and  $u_\alpha(x_0) = 0$ .

### 3.4 Proof of the theorem

For the time being we know that if  $u_0$  vanishes, it does so at a point of  $Z$ . Proving the theorem comes therefore to verify that if  $x_0$  is a point of  $Z$  then  $u_0(x_0) = 0$  *i.e.* the zero set of  $u_0$  coincides exactly with  $Z$ . Let then  $x_0 \in Z$  and let us show that  $u_\alpha(x_0) = 0, \forall \alpha > 0$ . We know that  $u_\alpha(x_0) \geq 0$ . Suppose  $u_\alpha(x_0) > 0$ , for some  $\alpha > 0$ . As  $u_\alpha$  is bounded and lipschitz continuous, it is almost everywhere differentiable and therefore one can find a sequence  $(x_n)_n$  such that

$$x_n \xrightarrow{n \rightarrow \infty} x_0 \quad \text{and} \quad Du_\alpha(x_n) \quad \text{exists} \quad \forall n.$$

Set  $p_n^\alpha = Du_\alpha(x_n)$ . Then for all  $n$ , there holds

$$\alpha u_\alpha(x_n) + H(x_n, p_n^\alpha) = 0 \tag{10}$$

Due to the boundedness of  $u_\alpha$  and the coercivity condition (3), the sequence  $(p_n^\alpha)_n$  is bounded so that it converges (up to a subsequence) to some  $p^\alpha \in \mathbb{R}^N$ . A passage to the limit in (10) then leads to

$$\alpha u_\alpha(x_0) + H(x_0, p^\alpha) = 0.$$

Now this is not possible because  $u_\alpha(x_0) > 0$  and by assumption  $H(x_0, p^\alpha) \geq 0$ . We thus have  $u_\alpha(x_0) = 0$  and so will be  $u_0(x_0)$  by passing to the limit in  $\alpha$ .

Now it remains to prove the Lemma 2 and we will be done.

### 3.5 Proof of Lemma 2

We know by [6] that there exists a unique  $\lambda$  for which 5 is solvable. Therefore, to prove the lemma, it suffices to prove that  $\lambda = 0$ . Consider for this sake, for some  $\alpha > 0$ , the unique solution  $u_\alpha$  of the equation

$$\alpha u_\alpha + H(x, Du_\alpha) = \lambda, \quad x \in \mathbb{R}^N. \quad (11)$$

Let  $x_0$  be a global minimum point of  $u_\alpha$ . By testing (11) with 0, one has  $\alpha u_\alpha(x_0) \geq \lambda$  and thus

$$\alpha u_\alpha(x) \geq \lambda, \quad \forall x \in \mathbb{R}^N. \quad (12)$$

Consider now the point  $x^*$  such that  $H(x^*, p) \geq 0$  for all  $p$ . As previously, we can find a sequence  $(x_n)_n$  which has  $x^*$  as limit and such that

$$\alpha u_\alpha(x_n) + H(x_n, Du_\alpha(x_n)) = \lambda$$

holds for all  $n$ . And thus, at the limit  $n$  goes to infinity, we recover

$$\alpha u_\alpha(x^*) + H(x^*, p^\alpha) = \lambda,$$

which leads to  $\alpha u_\alpha(x^*) \leq \lambda$ , which together with (12) gives  $\alpha u_\alpha(x^*) = \lambda$ . Now as  $\alpha$  was taken arbitrarily and  $(u_\alpha)_\alpha$  bounded, by letting  $\alpha$  go to zero, we obtain  $\lambda = 0$ .

## References

- [1] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, Springer-Verlag, Berlin, (1994).
- [2] M. Bostan and G. Namah, Time periodic viscosity solutions of Hamilton-Jacobi equations, *Comm. in Pure and Appl. Anal.*, **6**, 389-410 (2007).
- [3] A. Fathi, Sur la convergence du semi-groupe de Lax-Oleinik, *C. R. Acad. Sci. Paris, Sr. I Math.*, **327**, 267-270 (1998).
- [4] A. Fathi and A. Siconolfi, PDE aspects of Aubry-Mather theory for quasiconvex hamiltonians, *Calc. Var.* **22**, 185-228 (2005).
- [5] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi Equations*, Research Notes in Mathematics, Pitman, (1982).
- [6] P.-L. Lions, G. Papanicolaou and S.R.S. Varadhan, Homogenization of Hamilton-Jacobi equations, *Preprint*.

- [7] G. Namah and J.-M. Roquejoffre, Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations, *Comm. Partial Differential Equations*, **24**, 883–893 (1999).
- [8] J.-M. Roquejoffre, Comportement asymptotique des solutions d'equations de Hamilton-Jacobi monodimensionnelles, *C. R. Acad. Sci. Paris Sr. I Math.*, **326**, 185–189 (1998).